Almost Gorenstein rings of higher dimension

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AMS Special Session on Algebra and Representation theory

March 7, 2015

§1 History of almost Gorenstein rings

- In 1997, B. Barucci and R. Fröberg ([BF])
  $\cdots$ one-dimensional analytically unramified local rings
- In 2013, S. Goto, N. Matsuoka and T. T. Phuong ([GMP])
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  which are *not necessarily analytically unramified*.

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If it's possible, what's the definition of almost Gorenstein rings of higher dimension?

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Contents

§2 Almost Gorenstein local rings

Setting 2.1

- $(R, \mathfrak{m})$ a Cohen-Macaulay local ring with $d = \dim R$.
- $\exists$ the canonical module K_R.
- $|R/\mathfrak{m}| = \infty$.

Definition 2.2

We say that R is *an almost Gorenstein local ring*, if $\exists$ an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R-modules such that $\mu_R(C) = \mathrm{e}^0_{\mathfrak{m}}(C)$.

Therefore in Definition 2.2, if $C \neq (0)$, then C is Cohen-Macaulay and $\dim_R C = d - 1$. Moreover

$$\mu_R(C) = \mathrm{e}^0_{\mathfrak{m}}(C) \Longleftrightarrow \mathfrak{m}C = (f_2, f_3, \ldots, f_d)C$$

for some $f_2, f_3, \ldots, f_d \in \mathfrak{m}$.

Hence C is a *maximally generated Cohen-Macaulay module* in the sense of B. Ulrich (cf. [2]), which is called *an Ulrich R-module*.

Remark 2.3

Suppose that $d = 1$. Then TFAE.

(1) R is almost Gorenstein in the sense of Definition 2.2.

(2) R is almost Gorenstein in the sense of [GMP, Definition 3.1].

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Example 2.4

(1) $k[[t^3, t^4, t^5]]$.

(2) $k[[t^a, t^{a+1}, \ldots, t^{2a-3}, t^{2a-1}]]$ $(a \geq 4)$.

(3) $k[[X, Y, Z]]/(X, Y) \cap (Y, Z) \cap (Z, X)$.

(4) Suppose that R is not Gorenstein. If R is an almost Gorenstein local ring, then R is G-regular.

(5) 1-dimensional finite CM-representation type.

(6) 2-dimensional rational singularity.

Theorem 2.5 (NZD characterization)

(1) *If R is an almost Gorenstein local ring of dimension $d > 1$, then so is $R/(f)$ for genaral NZD $f \in \mathfrak{m}$.*

(2) *Let $f \in \mathfrak{m}$ be a NZD on R. If $R/(f)$ is an almost Gorenstein local ring, then so is R. When this is the case, $f \notin \mathfrak{m}^2$, if R is not Gorenstein.*

Corollary 2.6

Suppose that $d > 0$. If $R/(f)$ is an almost Gorenstein local ring for every NZD $f \in \mathfrak{m}$, then R is Gorenstein.

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Example 2.7

Let $U = k[[X_1, X_2, \ldots, X_n, Y_1, Y_2, , \ldots, Y_n]]$ $(n \geq 2)$ be the formal power series ring over an infinite field k and put

$$R = U/\mathrm{I}_2(\mathbb{M}),$$

where $\mathrm{I}_2(\mathbb{M})$ denotes the ideal of U generated by 2×2 minors of the matrix

$$\mathbb{M} = \begin{pmatrix} X_1 & X_2 & \cdots & X_n \\ Y_1 & Y_2 & \cdots & Y_n \end{pmatrix}.$$

Then R is almost Gorenstein with $\dim R = n + 1$ and $\mathrm{r}(R) = n - 1$.

Proof of Example 2.7.

Notice that

- $\{X_i - Y_{i-1}\}_{1 \le i \le n}$ (here $Y_0 = Y_n$) forms a regular sequence on R
- $R/(X_i - Y_{i-1} \mid 1 \le i \le n)R \cong k[[X_1, X_2, \ldots, X_n]]/\mathrm{I}_2(\mathbb{N}) = S$,

  where $\mathbb{N} = \begin{pmatrix} X_1 & X_2 & \cdots & X_{n-1} & X_n \\ X_2 & X_3 & \cdots & X_n & X_1 \end{pmatrix}$.

Then

- S is Cohen-Macaulay with $\dim S = 1$,
- $\mathfrak{n}^2 = x_1\mathfrak{n}$ and $\mathrm{K}_S \cong (x_1, x_2, \ldots, x_{n-1})$,

where $\mathfrak{n}$ is the maximal ideal of S, x_i is the image of X_i in S. Hence S is an almost Gorenstein local ring, since $\mathfrak{n}(x_1, x_2, \ldots, x_{n-1}) \subseteq (x_1)$. Thus R is almost Gorenstein. □

Theorem 2.8

Let $(S, \mathfrak{n})$ be a Noetherian local ring, $\varphi : R \to S$ a flat local homomorphism. Suppose that $S/\mathfrak{m}S$ is a RLR. Then TFAE.

(1) *R is an almost Gorenstein local ring.*

(2) *S is an almost Gorenstein local ring.*

Therefore

- R is almost Gorenstein $\Longleftrightarrow$ $R[[X_1, X_2, \ldots, X_n]]$ $(n \geq 1)$ is almost Gorenstein.
- R is almost Gorenstein $\Longleftrightarrow$ $\widehat{R}$ is almost Gorenstein.

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- R is almost Gorenstein $\Longleftrightarrow$ $\widehat{R}$ is almost Gorenstein.

§3 Semi-Gorenstein local rings

In this section we maintain Setting 2.1.

Definition 3.1

We say that R is *a semi-Gorenstein local ring*, if R is an almost Gorenstein local ring which possesses an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

such that either $C = (0)$, or $C \neq (0)$ and $C = \oplus_{i=1}^{\ell} C_i$ for some cyclic R-submodule C_i of C.

Therefore, if $C \neq (0)$, then

$$C_i \cong R/\mathfrak{p}_i \quad \text{for } \exists\ \mathfrak{p}_i \in \operatorname{Spec} R$$

such that $R/\mathfrak{p}_i$ is a RLR of dimension $d-1$.

Notice that

- almost Gorenstein local ring with $\dim R = 1$
- almost Gorenstein local ring with $\mathrm{r}(R) \leq 2$

are semi-Gorenstein.

Proposition 3.2

Let R be a semi-Gorenstein local ring. Then $R_\mathfrak{p}$ is semi-Gorenstein for $\forall \mathfrak{p} \in \operatorname{Spec} R$.

Therefore, if $C \neq (0)$, then

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Let R be a semi-Gorenstein local ring. Then $R_\mathfrak{p}$ is semi-Gorenstein for $\forall \mathfrak{p} \in \operatorname{Spec} R$.

Theorem 3.3

Let $(S, \mathfrak{n})$ *be a RLR,* $\mathfrak{a} \subsetneq S$ *an ideal of* S *with* $n = \mathrm{ht}_S\, \mathfrak{a}$*. Let* $R = S/\mathfrak{a}$*. Then TFAE.*

(1) R *is a semi-Gorenstein local ring, but not Gorenstein.*

(2) R *is Cohen-Macaulay,* $n \geq 2$*,* $r = \mathrm{r}(R) \geq 2$*, and* R *has a minimal* S*-free resolution of the form*:

$$0 \to F_n = S^r \xrightarrow{\mathbb{M}} F_{n-1} = S^q \to F_{n-2} \to \cdots \to F_1 \to F_0 = S \to R \to 0$$

where

$$ {}^t\mathbb{M} = \begin{pmatrix} y_{21}y_{22}\cdots y_{2\ell} & y_{31}y_{32}\cdots y_{3\ell} & \cdots & y_{r1}y_{r2}\cdots y_{r\ell} & z_1z_2\cdots z_m \\ x_{21}x_{22}\cdots x_{2\ell} & 0 & 0 & 0 & 0 \\ 0 & x_{31}x_{32}\cdots x_{3\ell} & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & x_{r1}x_{r2}\cdots x_{r\ell} & 0 \end{pmatrix},$$

$\ell = n+1$, $q \geq (r-1)\ell$, $m = q-(r-1)\ell$, *and* $x_{i1}, x_{i2}, \dots, x_{i\ell}$ *is a part of a regular system of parameters of* S *for* $2 \leq \forall i \leq r$.

When this is the case,

$$\mathfrak{a} = (z_1, z_2, \ldots, z_m) + \sum_{i=2}^{r} \mathrm{I}_2 \begin{pmatrix} y_{i1} & y_{i2} & \cdots & y_{i\ell} \\ x_{i1} & y_{i2} & \cdots & x_{i\ell} \end{pmatrix}.$$

Example 3.4

Let $\varphi : S = k[[X,Y,Z,W]] \longrightarrow R = k[[t^5,t^6,t^7,t^9]]$ be the k-algebra map defined by

$$\varphi(X) = t^5,\ \varphi(Y) = t^6,\ \varphi(Z) = t^7 \text{ and } \varphi(W) = t^9.$$

Then

$$0 \to S^2 \xrightarrow{\mathbb{M}} S^6 \to S^5 \to S \to R \to 0,$$

where

$${}^t\mathbb{M} = \begin{pmatrix} W & X^2 & XY & YZ & Y^2-XZ & Z^2-XW \\ X & Y & Z & W & 0 & 0 \end{pmatrix}.$$

Hence R is semi-Gorenstein with $\mathrm{r}(R) = 2$ and

$$\operatorname{Ker}\varphi = (Y^2 - XZ, Z^2 - XW) + \mathrm{I}_2\begin{pmatrix} W & X^2 & XY & YZ \\ X & Y & Z & W \end{pmatrix}.$$

§4 Almost Gorenstein graded rings

Setting 4.1

- $R = \bigoplus_{n \geq 0} R_n$ a Cohen-Macaulay graded ring with $d = \dim R$
- $(R_0, \mathfrak{m})$ a local ring
- $\exists$ the graded canonical module K_R
- $\mathfrak{M} = \mathfrak{m}R + R_+$
- $a = \mathrm{a}(R)$
- $|R/\mathfrak{m}| = \infty$

Definition 4.2

We say that R is *an almost Gorenstein graded ring*, if $\exists$ an exact sequence

$$0 \to R \to \mathrm{K}_R(-a) \to C \to 0$$

of graded R-modules such that $\mu_R(C) = \mathrm{e}^0_{\mathfrak{M}}(C)$.

Notice that

- R is an almost Gorenstein graded ring
  $\Longrightarrow R_{\mathfrak{M}}$ is an almost Gorenstein local ring.

Example 4.3

Let $R = k[X_1, X_2, \ldots, X_d]$ $(d \geq 1)$ be a polynomial ring over an infinite field k. Let $n \geq 1$ be an integer.

- $R^{(n)} = k[R_n]$ is an almost Gorenstein graded ring, if $d \leq 2$.
- Suppose that $d \geq 3$. Then $R^{(n)}$ is an almost Gorenstein graded ring if and only if $n \mid d$ or $d = 3$ and $n = 2$.

Theorem 4.4 (Goto-Iai [3])

Let A be a Gorenstein local ring, $I \subsetneq A$ an ideal of A. If $G = \mathrm{gr}_I(A)$ is an almost Gorenstein graded ring, then G is Gorenstein.

Theorem 4.5

Let $(A, \mathfrak{m})$ be a Gorenstein local ring of dimension $d \geq 3$ and Q a parameter ideal of A. Then TFAE.

(1) $\mathcal{R}(Q)$ *is an almost Gorenstein graded ring.*

(2) $Q = \mathfrak{m}$.

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(1) $\mathcal{R}(Q)$ *is an almost Gorenstein graded ring.*

(2) $Q = \mathfrak{m}$.

Our goal is the following.

Theorem 4.6

Let $(A, \mathfrak{m})$ *be a Cohen-Macaulay local ring with* $|A/\mathfrak{m}| = \infty$, $\exists \mathrm{K}_A$. *Let* I *be an* $\mathfrak{m}$*-primary ideal of* A.
If $G = \mathrm{gr}_I(A)$ *is an almost Gorenstein graded ring and* $\mathrm{r}(G) = \mathrm{r}(A)$, *then* A *is an almost Gorenstein local ring.*

Thank you very much for your attention.

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